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1987 J. Phys. A: Math. Gen. 20 189

(http://iopscience.iop.org/0305-4470/20/1/027)

J. Phys. A: Math. Gen. **20** (1987) 189-197. Printed in the UK

Renormalisation group calculations for a spin-1 Ising model with bilinear and biquadratic exchange interactions

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Received 20 January 1986, in final form 28 April 1986

Abstract. We perform mean-field and renormalisation group calculations for a spin-1 Ising model with bilinear and biquadratic exchange interactions. A special Baker-Hubbard formula is used to transform from discrete to continuous spin-like variables. In momentum space, this spin-1 model and the Ising metamagnet in zero staggered field can be described by the same reduced Hamiltonian. We show that the Gaussian tricritical fixed point cannot be reached without the inclusion of single-ion terms in the initial Hamiltonian.

1. Introduction

We report detailed mean-field and renormalisation group calculations for a spin-1 Ising model with bilinear and biquadratic exchange interactions. This model Hamiltonian, which is a special case of the Blume-Emery-Griffiths (BEG) model for the multicritical behaviour of ^{3}He-^{4}He mixtures (Blume *et al* 1971), is relevant, for instance, in the study of some very simple compressible Ising systems.

There are different ways to account for the influence of elastic vibrations on the critical properties of Ising spin systems (Salinas 1974, Bergman and Halperin 1976, Bruno and Sak 1980). For example, Domb (1956) considered an Ising model where the exchange parameter J depends on the volume of the crystal lattice. In Domb's model, there is a mechanical instability and the transition becomes first order. On the other hand, in a well known publication, Baker and Essam (1970) introduced a simple cubic Ising model where the exchange parameter is a linear function of the atomic displacements, the elastic potentials are harmonic and the shear forces are completely neglected. This compressible Ising model, which can be solved exactly in two dimensions, presents a continuous phase transition, with renormalised critical exponents at fixed densities. In a subsequent publication, Gunther *et al* (1971) noticed that the solution of the Baker-Essam model could have been considerably simplified in a particular ensemble, with fixed forces acting on all rows and columns of atoms. In this force ensemble, if we integrate the elastic degrees of freedom, it is easy to write an effective spin Hamiltonian

$$\mathcal{H} = -J_2 \sum_{(ij)} S_i S_j - J_4 \sum_{(ij)} S_i^2 S_j^2 \tag{1.1}$$

where J_2 is a linear function of the forces, J_4 is a positive constant and (ij) represents a sum over nearest neighbours on a hypercubic lattice in d dimensions. Of course, for spin $\frac{1}{2}$, the second term on the right-hand side of equation (1.1) is a constant and the problem reduces to the calculation of the usual Ising partition function. In this case, the transition is of second order in the force ensemble, with Ising critical exponents, and, as we remarked above, the critical indices corresponding to fixed densities are Fisher-renormalised (Fisher 1968). It is then interesting to investigate the nature of the phase transition of general spin compressible Ising models in the force ensemble. In the present paper, in particular, we assume the simplest possibility, namely $S_i = +1, 0, -1$, for all sites i, and focus the attention on the model Hamiltonian given by equation (1.1).

As we report in § 2, a naive mean-field calculation, via the Bogoliubov inequality with a one-parameter free trial Hamiltonian, leads to a tricritical point separating lines of second- and first-order phase transitions. By the way, this result still persists if we assume a continuous spin variable, insert the usual weight factor and perform an ε-expansion renormalisation group calculation. However, a more detailed mean-field calculation, involving a two-parameter trial Hamiltonian, indicates the suppression of the tricritical point and the line of first-order transitions, in qualitative agreement with results for the spin-$\frac{1}{2}$ model. Also, this is in agreement with exact calculations for the Curie–Weiss long-range version of the model Hamiltonian (1.1).

The lack of agreement between the mean-field calculations, as well as the possible occurrence of a fluctuation induced tricritical point (Aharony and Blankschtein 1984), motivated the undertaking of a renormalisation group analysis of the spin-1 Ising model given by equation (1.1). In § 3 we perform a Baker–Hubbard transformation (Baker 1962, Hubbard 1972) from discrete spins to a pair of continuous spin-like variables. To make contact with previous calculations (see, for example, Lawrie and Sarbach 1984), we include in equation (1.1) a single-ion term, given by $\Delta\,\Sigma_i\,S_i^2$. The reduced Hamiltonian, in terms of critical and non-critical spin fields, may be cast in the same form which had already been considered by Nelson and Fisher (1975) in the treatment of the Ising metamagnet. In § 4 we use the results of Nelson and Fisher's paper to reproduce the configuration of fixed points, with the inclusion of a Gaussian tricritical fixed point. As the Baker–Hubbard transformation does not require the use of unknown weight factors, we are able to show that, for $\Delta = 0$, the tricritical fixed point cannot be reached from the physical parameter space. We thus conclude that the transition of the Baker–Essam model is always second order, without the occurrence of fluctuation induced multicritical points. This is also in agreement with results from real space renormalisation group calculations for two-dimensional (Berker and Wortis 1976, Kaufman *et al* 1981) as well as three-dimensional (Yeomans and Fisher 1981) versions of the BEG model. Some final remarks and possible extensions of this work are presented in § 5.

2. Mean-field calculations

The mean-field expression for the Gibbs free energy, $G(T, H, N)$, where T is temperature, H is the applied field and N is the number of spins, may be obtained from the Bogoliubov inequality

$$G \leqslant G_0 + \langle \mathscr{H} - \mathscr{H}_0 \rangle_0 \equiv \Phi \tag{2.1}$$

where

$$G_0 = -k_B T \ln\left(\sum_{\{S_i\}} \exp(-\beta \mathcal{H}_0)\right) \tag{2.2}$$

$\beta = (k_B T)^{-1}$, k_B is Boltzmann's constant and $\mathcal{H}_0$ is a trial Hamiltonian. The sum is over spin configurations and the canonical average $\langle \ldots \rangle_0$ is taken with respect to $\mathcal{H}_0$. We have considered two distinct trial Hamiltonians:

$$\mathcal{H}_0 = -\eta \sum_i S_i \tag{2.3}$$

and

$$\mathcal{H}_0 = -\eta_1 \sum_i S_i - \eta_2 \sum_i S_i^2 \tag{2.4}$$

where η, η_1 and η_2 are variational parameters with respect to which we minimise Φ to obtain the mean-field approximation for the Gibbs free energy.

Using the trial Hamiltonian (2.3) we show the existence of a λ line which ends at a tricritical point given by $k_B T/qJ_2 = \frac{4}{3}$ and $p = J_4/J_2 = 3$. Using the trial Hamiltonian (2.4), with two variational parameters associated with S_i and S_i^2 respectively, as suggested by the renormalisation group calculations of the following section, we obtain

$$(1/N)\Phi = -(1/\beta)\ln[1 + 2\exp(\beta\eta_2)\cosh\beta\eta_1] - \tfrac{1}{2}qJ_2 m_1^2 - \tfrac{1}{2}qJ_4 m_2^2 - (H - \eta_1)m_1 + \eta_2 m_2 \tag{2.5}$$

where

$$m_1 = \frac{2\exp(\beta\eta_2)\sinh\beta\eta_1}{1 + 2\exp(\beta\eta_2)\cosh\beta\eta_1} \tag{2.6}$$

and

$$m_2 = \frac{2\exp(\beta\eta_2)\cosh\beta\eta_1}{1 + 2\exp(\beta\eta_2)\cosh\beta\eta_1}. \tag{2.7}$$

It is convenient to use equations (2.6) and (2.7) to write η_1 and η_2 in terms of m_1 and m_2. We then minimise Φ with respect to the non-critical density m_2 and write the Landau expansion

$$\frac{1}{N}\Phi = \frac{1}{\beta}\ln(1-a) + \tfrac{1}{2}qJ_4 a^2 - Hm_1 + \frac{1}{2\beta}\left(\frac{1}{a} - \beta qJ_2\right) m_1^2 + \frac{1}{\beta}\frac{-1 + 3a - 2\gamma a(1-a)}{24a^3[1 - \gamma a(1-a)]^2} m_1^4 + \mathrm{O}(m_1^6) \tag{2.8}$$

where $\gamma = \beta qJ_4$ and a is the solution of the equation

$$a = 2(1-a)\exp(\gamma a). \tag{2.9}$$

In the $T \times p$ space, the critical line is given by

$$a\beta qJ_2 = 1. \tag{2.10}$$

In this case, it is straightforward to see that there is no tricritical point, since the coefficient of the quartic term is positive along the critical line for all physical values of T and p. This conclusion still holds if we perform an exact calculation for the Curie-Weiss long-range version of Hamiltonian (1.1), as in the work of Tanaka and Mannari (1976). Also, it should be remarked that a Bragg-Williams calculation, as in the work of Lajzerowicz and Sivardière (1975), leads to the same mean-field equations of state which can be obtained from the minimisation of Φ.

3. The continuous variable formulation

To perform an ε-expansion renormalisation group calculation it is important to transform from discrete to continuous spin-like variables. This is usually done by the Gaussian transformation:

$$\exp\left(\frac{1}{2}\sum_{i,j} S_i K_{ij} S_j\right) = \text{constant} \times \prod_{k=1}^{N} \int_{-\infty}^{+\infty} \mathrm{d}X_k \exp\left(-\frac{1}{2}\sum_{i,j} X_i K_{ij}^{-1} X_j + \sum_i S_i X_i\right). \tag{3.1}$$

However, for most spin systems $K_{ii} = 0$ and this transformation is not well defined (since Tr $K = 0$, which leads to negative or zero eigenvalues for the matrix K). For spin $\frac{1}{2}$, $S_i^2 = 1$ and it is easy to remedy this problem. In this case we can add to K_{ij} an arbitrary constant matrix element, $c\delta_{i,j}$, the only consequence being a shift of the energy by a trivial constant term cN. For arbitrary spin, this trick does not work and we have to resort to another version of the Gaussian formula (Baker 1962):

$$\exp\left(\frac{1}{2}\sum_{i,j} S_i K_{ij} S_j\right) = \prod_{k=1}^{N} \int_{-\infty}^{+\infty} \frac{\mathrm{d}X_k}{(2\pi)^{1/2}} \exp\left(-\frac{1}{2}\sum_i X_i^2 - \sum_{i,j} X_i K_{ij}^{1/2} S_j\right) \tag{3.2}$$

which holds for all symmetric matrices K_{ij}, where $K_{ij}^{1/2}$ satisfies the equation

$$\sum_j K_{ij}^{1/2} K_{jk}^{1/2} = K_{ik}. \tag{3.3}$$

The partition function associated with Hamiltonian (1.1), in zero field but with the inclusion of a single-ion term, is given by

$$Z = \sum_{\{S_i\}} \exp\left(\frac{1}{2}\sum_{i,j} K_{ij} S_i S_j + \frac{1}{2}\sum_{i,j} L_{ij} S_i^2 S_j^2 - D\sum_i S_i^2\right) \tag{3.4}$$

where $(K_{ij};\, L_{ij}) \equiv (\beta J_2;\, \beta J_4)$, when i and j are nearest neighbours, and zero otherwise, and $D \equiv \beta\Delta$. Using the transformation (3.2) we have

$$Z = \sum_{\{S_i\}} \prod_k \int_{-\infty}^{+\infty} \frac{\mathrm{d}X_k}{(2\pi)^{1/2}} \frac{\mathrm{d}Y_k}{(2\pi)^{1/2}} \exp\left(-\frac{1}{2}\sum_i X_i^2 - \frac{1}{2}\sum_i Y_i^2 - \sum_{i,j} X_i K_{ij}^{1/2} S_j \right.$$

$$\left. - \sum_{i,j} Y_i L_{ij}^{1/2} S_j^2 - D\sum_i S_i^2\right) \tag{3.5}$$

where $K_{ij}^{1/2} = \beta J_2^{1/2}$ and $L_{ij}^{1/2} = \beta J_4^{1/2}$ when i and j are nearest neighbours. It should be remarked that the continuous variable X_i is associated with S_i and Y_i with S_i^2. After performing the sum over configurations we have

$$Z = \int \mathrm{D}X\, \mathrm{D}Y \exp\left(-\frac{1}{2}\sum_i X_i^2 - \frac{1}{2}\sum_i Y_i^2 + \sum_i \ln(1 + 2\,\mathrm{e}^{-w_i} \cosh x_i)\right) \tag{3.6}$$

where

$$w_i = \sum_j Y_j L_{ji}^{1/2} + D \qquad x_i = \sum_j X_j K_{ji}^{1/2}$$

and

$$\int \mathrm{D}X\, \mathrm{D}Y = \prod_{k=1}^{N} \int_{-\infty}^{+\infty} \frac{\mathrm{d}X_k}{(2\pi)^{1/2}} \frac{\mathrm{d}Y_k}{(2\pi)^{1/2}}. \tag{3.7}$$

The reduced Hamiltonian to perform the renormalisation group calculations may be found by expanding the integrand in equation (3.6) about its saddle point, given by the solutions of the equations

$$-X_i+\sum_j \frac{2\exp(-w_j)\sinh x_j}{1+2\exp(-w_j)\cosh x_j} K_{ij}^{1/2}=0 \tag{3.8}$$

and

$$-Y_i-\sum_j \frac{2\exp(-w_j)\cosh x_j}{1+2\exp(-w_j)\cosh x_j} L_{ij}^{1/2}=0. \tag{3.9}$$

There is a trivial paramagnetic solution, $X_i=0$ and $Y_i=Y_0$, such that

$$-Y_0=\frac{2\exp(-Y_0L_0^{1/2}-D)}{1+2\exp(-Y_0L_0^{1/2}-D)} L_0^{1/2} \tag{3.10}$$

where $L_0^{1/2}=\Sigma_j L_{ij}^{1/2}$ is the Fourier transform of $L_{ij}^{1/2}$ at zero momentum. It should be remarked that equations (3.8) and (3.9) correspond to the minimisation conditions of the mean-field free energy given by equation (2.5). The expansion about the paramagnetic solution yields the equation

$$Z=\exp(\bar{\mathscr{H}}_0)\int \mathrm{D}X\,\mathrm{D}Y\exp(\bar{\mathscr{H}}) \tag{3.11}$$

where

$$\bar{\mathscr{H}}_0=-\tfrac{1}{2}NY_0^2+N\ln[1+2\exp(-Y_0L_0^{1/2}-D)] \tag{3.12}$$

and $\bar{\mathscr{H}}$ is the reduced Hamiltonan. Then $\bar{\mathscr{H}}$ can be rewritten according to the following steps: (i) an expansion about the paramagnetic solution up to fourth-order terms in X_i and $\delta Y_i\equiv Y_i-Y_0$, (ii) a d-dimensional Fourier transformation, (iii) the usual low-momentum expansion of all the coefficients and (iv) a rescaling of the spin variables, $\sigma_{1,k}\to X_k$ and $\sigma_{2,k}\to\delta Y_k$. If we suppress the notation for vectors in the Fourier space, it is possible to write

$$\begin{aligned}\bar{\mathscr{H}}=&-\frac{1}{V}\sum_k \tfrac{1}{2}(r_1+k^2)\sigma_{1,k}\sigma_{1,-k}-\frac{1}{V}\sum_k \tfrac{1}{2}(r_2+k^2)\sigma_{2,k}\sigma_{2,-k}\\ &-\frac{1}{V^2}\sum_{k_1k_2}(\omega_1\sigma_{1,k_1}\sigma_{1,k_2}\sigma_{2,-k_1-k_2}-\omega_2\sigma_{2,k_1}\sigma_{2,k_2}\sigma_{2,-k_1-k_2})\\ &-\frac{1}{V^3}\sum_{k_1k_2k_3}(u_{11}\sigma_{1,k_1}\sigma_{1,k_2}\sigma_{1,k_3}\sigma_{1,-k_1-k_2-k_3}\\ &+2u_{12}\sigma_{1,k_1}\sigma_{1,k_2}\sigma_{2,k_3}\sigma_{2,-k_1-k_2-k_3}\\ &+u_{22}\sigma_{2,k_1}\sigma_{2,k_2}\sigma_{2,k_3}\sigma_{2,-k_1-k_2-k_3})\end{aligned} \tag{3.13}$$

where $r_1=k_B(T-T_1)/2A_{xx}J_2a^2$, $r_2=k_B(T-T_2)/2A_{yy}J_4a^2$, a is the lattice spacing, $Na^d=V$ and the remaining expressions are defined in the appendix.

4. The renormalisation group treatment

The reduced Hamiltonian, given by equation (3.13), corresponds to the Nelson and Fisher Hamiltonian for the Ising metamagnet in zero staggered field (compare with equation (3.3) of Nelson and Fisher (1975) for $r_{12}=\omega_3=\omega_4=0$). Some differences, such as the sign in front of ω_2, are associated with irrelevant variables, which will disappear after a few iterations. The renormalisation group analysis is therefore entirely analogous to that of Nelson and Fisher (1975). However, as this paper has a number of misprints, and there are no errata available, we have decided to reproduce the main steps of their calculations with the necessary corrections.

According to Nelson and Fisher (1975), we use perturbation theory to treat the non-quadratic terms of the reduced Hamiltonian, with the ansatz $r_1, u_{11}, u_{12}, u_{22}=O(\varepsilon)$ and $\omega_1, \omega_2=O(\sqrt{\varepsilon})$, where $\varepsilon=4-d$. In order to generate a new Hamiltonian $\bar{\mathscr{H}}'$ from $\bar{\mathscr{H}}$, we choose a rescaling factor $b>1$ and integrate over the spin variables $\sigma_{1,q}$, $\sigma_{2,q}$ with momentum bq outside the original Brillouin zone. The spin field rescaling factors c_1 and c_2 are allowed to be distinct. The inverse Feynman propagators are given by

$$G_1^{-1}(q, r_1)=r_1+e_1q^2 \tag{4.1a}$$

and

$$G_2^{-1}(q, r_2)=r_2+e_2q^2 \tag{4.1b}$$

where the reason to take $e_2\neq e_1$ (with $e_1=1$) will become clear later.

After each renormalisation group iteration, the cubic terms in the reduced Hamiltonian generate linear terms in the spin field $\sigma_{2,q}$. Then, to obtain a transformed Hamiltonian with the same form as before, without linear terms, it is necessary to shift σ_2 after each iteration. According to these procedures we obtain the recursion relations

$$r_1'=c_1^2b^{-d}[r_1+12A_{10}u_{11}-4A_{11}\omega_1^2-2A_{10}(\omega_1^2/r_2)+O(u_{12}, \omega_1\omega_2)] \tag{4.2}$$

$$r_2'=c_2^2b^{-d}(r_2-2A_{20}\omega_1^2+O(u_{12}, u_{22}, \omega_2^2, \omega_1\omega_2)) \tag{4.3}$$

$$e_1'=c_1^2b^{-d-2}e_1+O(\omega_1^2, \omega_1\omega_2) \tag{4.4}$$

$$e_2'=c_2^2b^{-d-2}e_2+O(\omega_2^2, \omega_1\omega_2) \tag{4.5}$$

$$\omega_1'=c_1^2c_2b^{-2d}(\omega_1-12A_{20}\omega_1u_{11}+4A_{21}\omega_1^3+O(\omega_1u_{12}, \omega_2u_{12}, \omega_1\omega_2^2)) \tag{4.6}$$

$$\omega_2'=c_2^3b^{-2d}(\omega_2+\tfrac{4}{3}A_{30}\omega_1^3+O(\omega_2u_{22}, \omega_1u_{12}, \omega_2^3, \omega_1u_{22})) \tag{4.7}$$

$$u_{11}'=c_1^4b^{-3d}(u_{11}-36A_{20}u_{11}^2+24A_{21}u_{11}\omega_1^2-4A_{22}\omega_1^4+O(u_{12}^2, u_{12}\omega_1^2)) \tag{4.8}$$

$$u_{12}'=c_1^2c_2^2b^{-3d}(u_{12}+24A_{30}u_{11}\omega_1^2-8A_{31}\omega_1^4+O(u_{12}u_{11}, u_{12}^2, u_{12}\omega_1^2, u_{22}\omega_1^2, \omega_1^2\omega_2^2)) \tag{4.9}$$

$$u_{22}'=c^4b^{-3d}(u_{22}-2A_{40}\omega_1^4+O(u_{22}^2, u_{12}^2, u_{12}\omega_1^2, u_{22}\omega_2^2, \omega_2^4)) \tag{4.10}$$

where

$$A_{lm}=\int_q^{>}[G_1(q)]^l[G_2(q)]^m \tag{4.11}$$

in which we integrate over the outer d-dimensional momentum shell as discussed before. We have also anticipated the fact that ω_2, u_{12} and u_{22} are irrelevant.

From equation (4.3) we see that if c_1 and c_2 are chosen such that e_1 and e_2 are kept equal to unity, then r_2 diverges when r_1 is at criticality. This happens because A_{20} contains two G_1 propagators which develop infrared singularities when $r_1=0$. This can be avoided by choosing

$$c_1=b^{3-\varepsilon/2}(1+O(\varepsilon^2)) \tag{4.12}$$

and

$$c_2 = b^{2-\varepsilon/2}[1 + A_{20}(\omega_1^2/r_2) + O(\varepsilon^2)]. \tag{4.13}$$

Also, r_2 is kept fixed, $e_2 = 1$ and we have

$$e_2' = b^{-2}(e_2 + O(\varepsilon^2)) \qquad \omega_2' = b^{-2+\varepsilon/2}(\omega_2 + O(\varepsilon^{3/2})) \tag{4.14a}$$

$$u_{22}' = b^{-4+\varepsilon}(u_{22} + O(\varepsilon^2)) \qquad u_{12}' = b^{-2+\varepsilon}(u_{12} + O(\varepsilon^2)) \tag{4.14b}$$

from which we see that e_2, ω_2, u_{22} and u_{12}, are irrelevant and disappear after a few renormalisation group iterations. Finally, we have, up to terms of $O(\varepsilon^2)$,

$$r_1' = b^2[r_1 + 12A_{10}u_{11} - 6A_{10}(\omega_1^2/r_2) - 12A_{20}r_1u_{11} + 6A_{20}r_1(\omega_1^2/r_2)] \tag{4.15}$$

$$\omega_1' = \omega_1 + \omega_1\tfrac{1}{2}\varepsilon \ln b - 12A_{20}\omega_1u_{11} + 5A_{20}\omega_1^3/r_2 \tag{4.16}$$

$$u_{11}' = u_{11} + u_{11}\varepsilon \ln b - 36A_{20}u_{11}^2 + 24A_{20}u_{11}\omega_1/r_2 - 4A_{20}\omega_1^4/r_2^2 \tag{4.17}$$

where the integrals A_{l0} are calculated for $d = 4$ with $r_1 = 0$. At this point, it should be emphasised that we are considering the case $r_2 > r_1$. This corresponds to $T_2 < T_1$, i.e. either to $\Delta = 0$ or to $p \equiv J_4/J_2 < 1$ for $\Delta \neq 0$, as can be seen from equations (3.10) and (A4).

If we define

$$x = \omega_1^2/r_2 \tag{4.18}$$

the fixed points of equations (4.16) and (4.17) are given by

$$u_{11}^* = 0 \qquad x^* = 0 \tag{4.19a}$$

$$u_{11}^* = \tfrac{1}{36}\bar{\varepsilon} \qquad x^* = 0 \tag{4.19b}$$

$$u_{11}^* = \tfrac{1}{9}\bar{\varepsilon} \qquad x^* = \tfrac{1}{6}\bar{\varepsilon} \tag{4.19c}$$

$$u_{11}^* = \tfrac{1}{4}\bar{\varepsilon} \qquad x^* = \tfrac{1}{2}\bar{\varepsilon} \tag{4.19d}$$

where

$$\bar{\varepsilon} = \frac{\varepsilon \ln b}{A_{20}(r_1 = 0,\, e_1 = 1)} = c\varepsilon. \tag{4.20}$$

As $A_{20} \sim \ln b$, the constant c does not depend on b. The nature of the fixed points can be determined by studying the eigenvalues $\Lambda_i = b^{\lambda_i}$ of the linearised recursion relations

$$\begin{pmatrix} r_1' - r_1^* \\ u_{11}' - u_{11}^* \\ x' - x^* \end{pmatrix} = L \begin{pmatrix} r_1 - r_1^* \\ u_{11} - u_{11}^* \\ x - x^* \end{pmatrix}. \tag{4.21}$$

In particular, the critical exponent ν is given by $\nu = 1/\lambda_1$, where λ_1 is the largest eigenvalue. From equations (4.4) and (4.12), we see that $e_1 = 1 + O(\varepsilon^2)$, which yields $\eta = O(\varepsilon^2)$. These results are summarised in table 1.

It remains to be analysed whether the fixed points are accessible from the physical parameter space. The condition for the existence of a tricritical point is the vanishing

Table 1. Fixed points, eigenvalues λ_i and the corresponding eigenvectors y_i in the subspace u_{11}, x and the critical exponents ν associated with the recursion relations given by equations (4.15)–(4.17).

Fixed points	u_{11}^*	x^*	r^*	λ_1	λ_2	y_2	λ_3	y_3	ν	Type
(a)	0	0	0	2	ε	(1, 0)	ε	(0, 1)	$\frac{1}{2}$	Gaussian
(b)	$\frac{1}{36}\bar{\varepsilon}$	0	$-\dfrac{\bar{\varepsilon}b^2A_{10}}{3(b^2-1)}$	$2-\frac{1}{3}\varepsilon$	$-\varepsilon$	(1, 0)	$\frac{1}{3}\varepsilon$	(1, 2)	$\frac{1}{2}+\frac{1}{12}\varepsilon$	Ising-like
(c)	$\frac{1}{9}\bar{\varepsilon}$	$\frac{1}{6}\bar{\varepsilon}$	$-\dfrac{\bar{\varepsilon}b^2A_{10}}{3(b^2-1)}$	$2-\frac{1}{3}\varepsilon$	$-\frac{1}{3}\varepsilon$	(1, 3)	$-\varepsilon$	(2, 3)	$\frac{1}{2}+\frac{1}{12}\varepsilon$	Ising-like
(d)	$\frac{1}{4}\bar{\varepsilon}$	$\frac{1}{2}\bar{\varepsilon}$	0	2	$-\varepsilon$	(1, 2)	ε	(1, 3)	$\frac{1}{2}$	Gaussian

of the coefficient g of the quartic term in the critical field σ_1 after the integration over the non-critical fields. It is easy to show that

$$g = u_{11} - \omega_1^2/2r_2. \tag{4.22}$$

Using the expressions for the couplings, given in § 3, we have

$$\begin{aligned} g &= \frac{a^{d-4}d^2}{A_{xx}^2}\left(A_{xxxx} - \frac{A_{xxy}^2 dJ_4}{k_B(T-T_2)}\right) \\ &= \frac{a^{d-4}d^2(1+2\,e^{\tilde{p}})(2\,e^{\tilde{p}}-p-\frac{1}{2})}{6\,e^{\tilde{p}}(2\,e^{\tilde{p}}-p+1)} \end{aligned} \tag{4.23}$$

where $p \equiv -Y_0L_0^{1/2}$ and $\tilde{p} \equiv -w_0 = p - D$. For $D=0$, as in the Baker–Essam model, $g \neq 0$ for all p, and there is no tricritical point. For $D \neq 0$, however, as in the work of Blume *et al* (1971), g may vanish and there is a tricritical point. It should be remarked that the parameter p may be easily interpreted as the ratio J_4/J_2 up to terms of order ε.

5. Conclusions

We have performed mean-field and renormalisation group calculations for a spin-1 Ising model with bilinear and biquadratic exchange interactions. Although a naive mean-field variational approach leads to a tricritical point, a more refined approximation, based on two variational parameters, shows that the transition is always second order.

We have used a special Baker–Hubbard Gaussian formula to transform the Hamiltonian, with the addition of single-ion terms, from discrete to continuous spin-like variables. As the transformation keeps track of the original parameters of the model, we were able to establish the true multicritical behaviour. The reduced Hamiltonian for the spin-1 model is identical to the Hamiltonian of Nelson and Fisher's metamagnet in zero staggered field. The Gaussian tricritical fixed point, however, cannot be reached if we start from a Hamiltonian with bilinear and biquadratic exchange interactions only. On the other hand, we do have a tricritical point, in agreement with previous calculations, if the initial Hamiltonian includes single-ion terms. In a forthcoming publication we plan to use the Baker–Hubbard transformation to consider the complete Blume–Emery–Griffiths model, with the inclusion of cubic terms, and to make contact with the real space renormalisation group calculations.

Appendix

Explicit forms of the coefficients in equation (3.13) for the reduced Hamiltonian:

$$w_0 = Y_0 L_0^{1/2} + D \tag{A1}$$

$$T_1 = 4A_{xx}J_2 d/k_B \qquad T_2 = 4A_{yy}J_4 d/k_B \tag{A2}$$

$$\omega_1 = a^{d/2-3}A_{xxy}d^{3/2}/A_{xx}A_{yy}^{1/2} \qquad \omega_2 = a^{d/2-3}A_{yyy}d^{3/2}/A_{yy}^{3/2}$$

$$u_{11} = a^{d-4}A_{xxxx}d^2/A_{xx}^2 \qquad u_{22} = a^{d-4}A_{yyyy}d^2/A_{yy}^2 \tag{A3}$$

$$2u_{12} = a^{d-4}A_{xxyy}d^2/A_{xx}A_{yy}$$

$$A_{xx} = -Y_0/2L_0^{1/2} \qquad A_{yy} = \frac{Y_0^2 \exp(w_0)}{4(L_0^{1/2})^2} \qquad A_{xxy} = \frac{Y_0^2 \exp(w_0)}{4(L_0^{1/2})^2}$$

$$A_{yyy} = -\frac{Y_0^3 \exp(w_0)[2-\exp(w_0)]}{24(L_0^{1/2})^3} \qquad A_{xxxx} = -\frac{Y_0^3[8+2\exp(w_0)-\exp(2w_0)]}{96(L_0^{1/2})^3} \tag{A4}$$

$$A_{xxyy} = -\frac{Y_0^3 \exp(w_0)[2-\exp(w_0)]}{16(L_0^{1/2})^3} \qquad A_{yyyy} = \frac{Y_0^4 \exp(w_0)[8\exp(w_0)-4-\exp(2w_0)]}{192(L_0^{1/2})^4}.$$

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